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He's frequency formulation for nonlinear oscillators using a golden mean location

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ABSTRACT

He's frequency formulation, derived on the basis of an ancient Chinese mathematical method, is an effective method for treating nonlinear oscillators. There are different approaches for choosing location points in the formulation. This paper suggests another location point, an example is given, and the accuracy is improved.

1. Introduction

An ancient Chinese mathematical method was outlined in a review article [1] and used to treat nonlinear oscillators; also a frequency formulation was proposed, where the location point was chosen as $t = 0$. Geng and Cai suggested another choice for the location point: $t = T/N$, where T is the period of the oscillator, and N is generally chosen as $N = 12$. The formulation was further developed by the originator in Refs. [2–4]. Due to the simplicity of the formulation, various kinds of nonlinear oscillators can be easily obtained; see Refs. [5–10]. Another ancient Chinese method, called the He Chengtian inequality method, was also developed to solve nonlinear oscillators; see Refs. [11–13]. Zhong [14] and Lan and Yang [15] asserted that some complex nonlinear equations can be easily solved by ancient Chinese mathematical methods. Though much progress has been made in the application of ancient Chinese mathematics to modern technology, some open problems await a perfect answer. This paper will show the problem, and suggest a new idea for the choice of the location point.

2. The ancient Chinese method

Below is an algebraic equation:

$$f(x) = 0. \tag{1}$$

If we want to find its root, let x_1 and x_2 be approximate ones, which lead to the remainders $f(x_1)$ and $f(x_2)$ respectively. According to the ancient Chinese method, the approximate solution of the equation is

$$x = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}. \tag{2}$$

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3. He's frequency formulation

Consider the following nonlinear oscillator:

$$u'' + \frac{u}{1+\varepsilon u^2} = 0, \qquad u(0) = A, \qquad u'(0) = 0. \tag{3}$$

This equation can be solved by the energy balance method [16–18], the parameter-expansion method [19], and the homotopy perturbation method [20].

According to He's formulation [1–4], we substitute two trial functions $u_1(t) = A\cos t$ and $u_2(t) = A\cos\omega t$ into Eq. (3); these are respectively the solutions of the following two linear equations:

$$u'' + u = 0 \tag{4}$$

and

$$u'' + \omega^2 u = 0. \tag{5}$$

Then we get the two residuals

$$R_1(t) = \frac{A\cos t}{1+\varepsilon A^2\cos^2 t} - A\cos t \tag{6}$$

$$R_2(t) = \frac{A\cos\omega_t}{1+\varepsilon A^2\cos^2\omega_t} - \omega^2 A\cos\omega_t. \tag{7}$$

According to Geng and Cai's modification [21], the approximate frequency can be obtained as follows:

$$\omega_H^2 = \frac{R_2\,(t_2) - \omega^2 R_1\,(t_1)}{R_2\,(t_2) - R_1\,(t_1)} \tag{8}$$

where t_1 and t_2 are location points. The phase of the residuals is determined by the location points.

In previous research [1], the phase of the residuals was chosen as 0 for simplicity, i.e.,

$$t_1 = 0 \qquad t_2 = 0. \tag{9}$$

The residuals are as follows:

$$R_1(0) = \frac{A}{1+\varepsilon A^2} - A \tag{10}$$

$$R_2(0) = \frac{A}{1+\varepsilon A^2} - \omega^2 A. \tag{11}$$

According to Eqs. (10) and (11), the frequency formulation is

$$\begin{aligned}\omega_{H1}^2 &= \frac{R_2(0) - \omega^2 R_1\,(0)}{R_2(0) - R_1(0)}\\ &= \frac{\left(\frac{A}{1+\varepsilon A^2} - \omega^2 A\right) - \omega^2\left(\frac{A}{1+\varepsilon A^2} - A\right)}{\left(\frac{A}{1+\varepsilon A^2} - \omega^2 A\right) - \left(\frac{A}{1+\varepsilon A^2} - A\right)}\\ &= \frac{1}{1+\varepsilon A^2}.\end{aligned} \tag{12}$$

Geng and Cai [21] suggested

$$t_1 = \frac{T_1}{12}, \qquad t_2 = \frac{T_2}{12}. \tag{13}$$

Hereby T_1 and T_2 are respectively the periods of the two linear equations $u'' + u = 0$ and $u'' + \omega^2 u = 0$, and the phase of the residuals is $\pi/6$. The frequency formulation becomes

$$\begin{aligned}\omega_{H2}^2 &= \frac{R_2\left(\frac{T_2}{12}\right) - \omega^2 R_1\left(\frac{T_1}{12}\right)}{R_2\left(\frac{T_2}{12}\right) - R_1\left(\frac{T_1}{12}\right)}\\ &= \frac{1}{1+\frac{3}{4}\varepsilon A^2}\end{aligned} \tag{14}$$

which is the same as that obtained by perturbation methods [22].

4. New phases of residuals

Now we suggest a new phase of the residuals through setting the phase equal to $0.382 \times \frac{\pi}{2} = 0.191\pi$, a golden mean location of $\pi/2$. We set

$$t_1 = 0.0955T_1, \qquad t_2 = 0.0955T_2. \tag{15}$$

The frequency can be obtained; it is

$$\begin{aligned}\omega_{H3}^2 &= \frac{R_2(0.0955T_2) - \omega^2 R_1(0.0955T_1)}{R_2(0.0955T_2) - R_1(0.0955T_1)}\\ &= \frac{1}{1 + 0.6811\varepsilon A^2}.\end{aligned} \tag{16}$$

To illustrate the accuracies of the three approximate results obtained above, we compare their corresponding periods:

$$T_{H1} = \frac{2\pi}{\omega_{H1}} = 2\pi\sqrt{1 + \varepsilon A^2} \tag{17}$$

$$T_{H2} = \frac{2\pi}{\omega_{H2}} = 2\pi\sqrt{1 + \frac{3}{4}\varepsilon A^2} \tag{18}$$

$$T_{H3} = \frac{2\pi}{\omega_{H3}} = 2\pi\sqrt{1 + 0.6811\varepsilon A^2} \tag{19}$$

with the exact one [22]:

$$T_{\text{ex}} = 4\sqrt{\varepsilon}\int_0^A \frac{\mathrm{d}u}{\sqrt{\ln\left(1 + \varepsilon A^2\right) - \ln\left(1 + \varepsilon u^2\right)}}. \tag{20}$$

In the case $\varepsilon A^2 \to \infty$, we have

$$\lim_{\varepsilon A^2 \to \infty} T_{\text{ex}} = 4\sqrt{\varepsilon}\int_0^A \frac{\mathrm{d}u}{\sqrt{2(\ln A - \ln u)}} = 2\sqrt{2\pi\varepsilon}A \tag{21}$$

and

$$\lim_{\varepsilon A^2 \to \infty} \frac{T_{H1}}{T_{\text{ex}}} = \frac{2\pi \cdot \sqrt{\varepsilon}A}{2\sqrt{2\pi\varepsilon}A} = 1.2533 \tag{22}$$

$$\lim_{\varepsilon A^2 \to \infty} \frac{T_{H2}}{T_{\text{ex}}} = \frac{2\pi \cdot \frac{\sqrt{3}}{2}\sqrt{\varepsilon}A}{2\sqrt{2\pi\varepsilon}A} = 1.0854 \tag{23}$$

$$\lim_{\varepsilon A^2 \to \infty} \frac{T_{H3}}{T_{\text{ex}}} = \frac{2\pi \cdot \sqrt{0.6811\varepsilon}A}{2\sqrt{2\pi\varepsilon}A} = 1.0343. \tag{24}$$

The accuracy is greatly improved.

5. Summary

In this work we suggest an alternative approach to the choice of the location point in a frequency formulation.

References

[1] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.
[2] J.H. He, Comment on 'He's frequency formulation for nonlinear oscillators', European J. Phys. 29 (4) (2008) L19–L22.
[3] J.H. He, An improved amplitude–frequency formulation for nonlinear oscillators, Int. J. Nonlinear Sci. Numer. Simul. 9 (2008) 211–212.
[4] J.H. He, An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering, Internat. J. Modern Phys. B 22 (2008) 3487–3578.
[5] X.C. Cai, W.Y. Wu, He's frequency formulation for the relativistic harmonic oscillator, Comput. Math. Appl. 58 (2009) 2358–2359.
[6] H.L. Zhang, Application of He's amplitude–frequency formulation to a nonlinear oscillator with discontinuity, Comput. Math. Appl. 58 (2009) 2197–2198.
[7] Y.N. Zhang, F. Xu, L.L. Deng, Exact solution for nonlinear Schrodinger equation by He's frequency formulation, Comput. Math. Appl. 58 (2009) 2449–2451.
[8] J. Fan, He's frequency–amplitude formulation for the Duffing harmonic oscillator, Comput. Math. Appl. 58 (2009) 2473–2476.
[9] L. Zhao, He's frequency–amplitude formulation for nonlinear oscillators with an irrational force, Comput. Math. Appl. 58 (2009) 2477–2479.

[10] Z.F. Ren, G.Q. Liu, Y.X. Kang, et al., Application of He's amplitude–frequency formulation to nonlinear oscillators with discontinuities, Phys. Scr. 80 (2009) 045003.
[11] J.H. He, Max–min approach to nonlinear oscillators, Int. J. Nonlinear Sci. Numer. Simul. 9 (2008) 207–210.
[12] S.A. Demirbag, M.O. Kaya, Application of He's max–min approach to a generalized nonlinear discontinuity equation, Int. J. Nonlinear Sci. Numer. Simul. 11 (2010) 269–272.
[13] D.Q. Zeng, Y.Y. Lee, Analysis of strongly nonlinear oscillator using the max–min approach, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 1361–1368.
[14] T. Zhong, Ancient Chinese musical scales: best approximations, but why? Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 161–166.
[15] J.C. Lan, Z. Yang, Continued fraction method for an ancient Chinese musical equation, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 167–169.
[16] H.L. Zhang, Periodic solutions for some strongly nonlinear oscillations by He's energy balance method, Comput. Math. Appl. 58 (2009) 2480–2485.
[17] H.L. Zhang, Y.G. Xu, J.R. Chang, Application of He's energy balance method to a nonlinear oscillator with discontinuity, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 207–214.
[18] G.A. Afrouzi, D.D. Ganji, R.A. Talarposhti, He's energy balance method for nonlinear oscillators with discontinuities, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 301–304.
[19] S.T. Mohyud-Din, M.A. Noor, K.I. Noor, Parameter-expansion techniques for strongly nonlinear oscillators, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 581–583.
[20] S.S. Ganji, D.D. Ganji, S. Karimpour, et al., Applications of He's homotopy perturbation method to obtain second-order approximations of the coupled two-degree-of-freedom systems, Int. J. Nonlinear Sci. Numer. Simul. 10 (2009) 305–314.
[21] L. Geng, X.C. Cai, He's frequency formulation for nonlinear oscillators, European J. Phys. 28 (2007) 923–931.
[22] J.H. He, Non-perturbative methods for strongly nonlinear problems, dissertation, de-Verlag im Internet GmbH, Berlin, 2006, pp. 42–43.